1.24 Consider solution of the two-dimensional Poisson equation problem of Problem 1.21, a unit square with zero potential on the boundary and a constant unit charge density in the interior, by the technique of relaxation. Choose $h = 0.25$ so that there are nine interior sites. Use symmetry to reduce the number of needed sites to three, at (0.25, 0.25), (0.5, 0.25), and (0.5, 0.5). With so few sites, it is easy to do the iterations with a block of paper and a pocket calculator, but suit yourself.

(a) Use the "improved grid" averaging of Problem 1.22 and the simple (Jacobian) iteration scheme, starting with $\Phi = 1.0$ at all three interior sites. Do at least six iterations, preferably eight or ten.

(b) Repeat the iteration procedure with the same starting values, but using Gauss–Seidel iteration.

(c) Graph the two sets of results of each iteration versus iteration number and compare with the exact values, $4\pi\epsilon_0\Phi(0.25, 0.25) = 0.5691$, $4\pi\epsilon_0\Phi(0.5, 0.25) = 0.7205$, $4\pi\epsilon_0\Phi(0.5, 0.5) = 0.9258$. Comment on rate of convergence and final accuracy.

CHAPTER 2

Boundary-Value Problems in Electrostatics: I

Many problems in electrostatics involve boundary surfaces on which either the potential or the surface-charge density is specified. The formal solution of such problems was presented in Section 1.10, using the method of Green functions. In practical situations (or even rather idealized approximations to practical situations) the discovery of the correct Green function is sometimes easy and sometimes not. Consequently a number of approaches to electrostatic boundary-value problems have been developed, some of which are only remotely connected to the Green function method. In this chapter we will examine three of these special techniques: (1) the method of images, which is closely related to the use of Green functions; (2) expansion in orthogonal functions, an approach directly through the differential equation and rather remote from the direct construction of a Green function; (3) an introduction to finite element analysis (FEA), a broad class of numerical methods. A major omission is the use of complex-variable techniques, including conformal mapping, for the treatment of two-dimensional problems. The topic is important, but lack of space and the existence of self-contained discussions elsewhere accounts for its absence. The interested reader may consult the references cited at the end of the chapter.

2.1 *Method of Images*

The method of images concerns itself with the problem of one or more point charges in the presence of boundary surfaces, for example, conductors either grounded or held at fixed potentials. Under favorable conditions it is possible to infer from the geometry of the situation that a small number of suitably placed charges of appropriate magnitudes, external to the region of interest, can simulate the required boundary conditions. These charges are called *image charges*, and the replacement of the actual problem with boundaries by an enlarged region with image charges but not boundaries is called the *method of images*. The image charges must be external to the volume of interest, since their potentials must be solutions of the Laplace equation inside the volume; the "particular integral" (i.e., solution of the Poisson equation) is provided by the sum of the potentials of the charges inside the volume.

A simple example is a point charge located in front of an infinite plane conductor at zero potential, as shown in Fig. 2.1. It is clear that this is equivalent to the problem of the original charge and an equal and opposite charge located at the mirror-image point behind the plane defined by the position of the conductor.

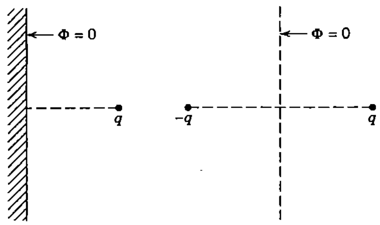

Figure 2.1 Solution by method of images. The original potential problem is on the left, the equivalent-image problem on the right.

2.2 *Point Charge in the Presence of a Grounded Conducting Sphere*

As an illustration of the method of images we consider the problem illustrated in Fig. 2.2 of a point charge q located at $\mathbf{y}$ relative to the origin, around which is centered a grounded conducting sphere of radius a. We seek the potential $\Phi(\mathbf{x})$ such that $\Phi(|\mathbf{x}| = a) = 0$. By symmetry it is evident that the image charge q' (assuming that only one image is needed) will lie on the ray from the origin to the charge q. If we consider the charge q *outside* the sphere, the image position $\mathbf{y}'$ will lie inside the sphere. The potential due to the charges q and q' is:

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{|\mathbf{x} - \mathbf{y}|} + \frac{q'/4\pi\epsilon_0}{|\mathbf{x} - \mathbf{y}'|} \tag{2.1}$$

We now must try to choose q' and $|\mathbf{y}'|$ such that this potential vanishes at $|\mathbf{x}| = a$. If $\mathbf{n}$ is a unit vector in the direction $\mathbf{x}$, and $\mathbf{n}'$ a unit vector in the direction $\mathbf{y}$, then

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{|x\mathbf{n} - y\mathbf{n}'|} + \frac{q'/4\pi\epsilon_0}{|x\mathbf{n} - y'\mathbf{n}'|} \tag{2.2}$$

If x is factored out of the first term and y' out of the second, the potential at $x = a$ becomes:

$$\Phi(x = a) = \frac{q/4\pi\epsilon_0}{a\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right|} + \frac{q'/4\pi\epsilon_0}{y'\left|\mathbf{n}' - \frac{a}{y'}\mathbf{n}\right|} \tag{2.3}$$

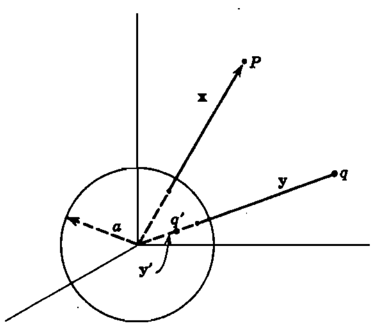

Figure 2.2 Conducting sphere of radius a, with charge q and image charge q'.

From the form of (2.3) it will be seen that the choices:

$$\frac{q}{a} = -\frac{q'}{y'}, \qquad \frac{y}{a} = \frac{a}{y'}$$

make $\Phi(x = a) = 0$, for all possible values of $\mathbf{n} \cdot \mathbf{n}'$. Hence the magnitude and position of the image charge are

$$q' = -\frac{a}{y}q, \qquad y' = \frac{a^2}{y} \tag{2.4}$$

We note that, as the charge q is brought closer to the sphere, the image charge grows in magnitude and moves out from the center of the sphere. When q is just outside the surface of the sphere, the image charge is equal and opposite in magnitude and lies just beneath the surface.

Now that the image charge has been found, we can return to the original problem of a charge q outside a grounded conducting sphere and consider various effects. The actual charge density induced on the surface of the sphere can be calculated from the normal derivative of Φ at the surface:

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial x}\right|_{x=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1 - \frac{a^2}{y^2}}{\left(1 + \frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma\right)^{3/2}} \tag{2.5}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{y}$. This charge density in units of $-q/4\pi a^2$ is shown plotted in Fig. 2.3 as a function of γ for two values of y/a. The concentra-

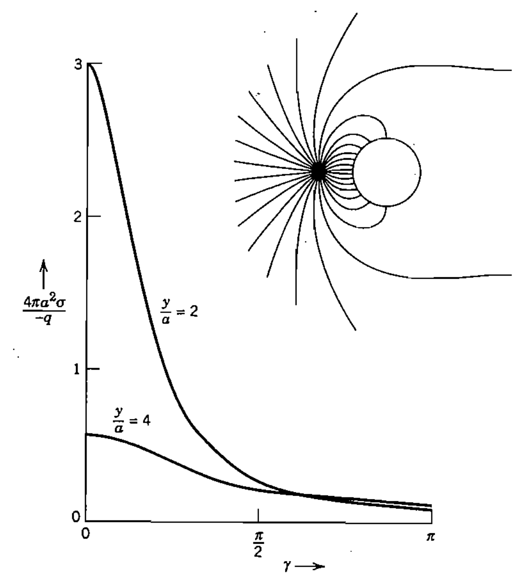

Figure 2.3 Surface-charge density σ induced on the grounded sphere of radius a as a result of the presence of a point charge q located a distance y away from the center of the sphere. σ is plotted in units of $-q/4\pi a^2$ as a function of the angular position γ away from the radius to the charge for $y = 2a$, $4a$. Inset shows lines of force for $y = 2a$.

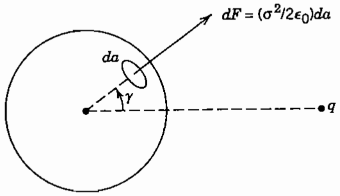

Figure 2.4

tion of charge in the direction of the point charge q is evident, especially for $y/a = 2$. It is easy to show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge, as it must be, according to Gauss's law.

The force acting on the charge q can be calculated in different ways. One (the easiest) way is to write down immediately the force between the charge q and the image charge q'. The distance between them is $y - y' = y(1 - a^2/y^2)$. Hence the attractive force, according to Coulomb's law, is:

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2} \tag{2.6}$$

For large separations the force is an inverse cube law, but close to the sphere it is proportional to the inverse square of the distance away from the surface of the sphere.

The alternative method for obtaining the force is to calculate the total force acting on the surface of the sphere. The force on each element of area da is $(\sigma^2/2\epsilon_0)\,da$, where σ is given by (2.5), as indicated in Fig. 2.4. But from symmetry it is clear that only the component parallel to the radius vector from the center of the sphere to q contributes to the total force. Hence the total force acting on the sphere (equal and opposite to the force acting on q) is given by the integral:

$$|\mathbf{F}| = \frac{q^2}{32\pi^2\epsilon_0 a^2}\left(\frac{a}{y}\right)^2\left(1 - \frac{a^2}{y^2}\right)^2 \int \frac{\cos\gamma}{\left(1 + \frac{a^2}{y^2} - \frac{2a}{y}\cos\gamma\right)^3}\,d\Omega \tag{2.7}$$

Integration immediately yields (2.6).

The whole discussion has been based on the understanding that the point charge q is *outside* the sphere. Actually, the results apply equally for the charge q *inside* the sphere. The only change necessary is in the surface-charge density (2.5), where the normal derivative out of the conductor is now radially inward, implying a change in sign. The reader may transcribe all the formulas, remembering that now $y \le a$. The angular distributions of surface charge are similar to those of Fig. 2.3, but the total induced surface charge is evidently equal to $-q$, independent of y.

2.3 *Point Charge in the Presence of a Charged, Insulated, Conducting Sphere*

In the preceding section we considered the problem of a point charge q near a grounded sphere and saw that a surface-charge density was induced on the sphere. This charge was of total amount $q' = -aq/y$, and was distributed over the surface in such a way as to be in equilibrium under all forces acting.

If we wish to consider the problem of an insulated conducting sphere with total charge Q in the presence of a point charge q, we can build up the solution for the potential by linear superposition. In an operational sense, we can imagine that we start with the grounded conducting sphere (with its charge q' distributed over its surface). We then disconnect the ground wire and add to the sphere an amount of charge $(Q - q')$. This brings the total charge on the sphere up to Q. To find the potential we merely note that the added charge $(Q - q')$ will distribute itself *uniformly* over the surface, since the electrostatic forces due to the point charge q are already balanced by the charge q'. Hence the potential due to the added charge $(Q - q')$ will be the same as if a point charge of that magnitude were at the origin, at least for points outside the sphere.

The potential is the superposition of (2.1) and the potential of a point charge $(Q - q')$ at the origin:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x} - \mathbf{y}|} - \frac{aq}{y\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Q + \frac{a}{y}q}{|\mathbf{x}|}\right] \tag{2.8}$$

The force acting on the charge q can be written down directly from Coulomb's law. It is directed along the radius vector to q and has the magnitude:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2 - a^2)}{y(y^2 - a^2)^2}\right]\frac{\mathbf{y}}{y} \tag{2.9}$$

In the limit of $y \gg a$, the force reduces to the usual Coulomb's law for two small charged bodies. But close to the sphere the force is modified because of the induced charge distribution on the surface of the sphere. Figure 2.5 shows the force as a function of distance for various ratios of Q/q. The force is expressed in units of $q^2/4\pi\epsilon_0 y^2$; positive (negative) values correspond to a repulsion (attraction). If the sphere is charged oppositely to q, or is uncharged, the force is attractive at all distances. Even if the charge Q is the same sign as q, however, the force becomes attractive at very close distances. In the limit of $Q \gg q$, the point of zero force (unstable equilibrium point) is very close to the sphere, namely, at $y \simeq a(1 + \frac{1}{2}\sqrt{q/Q})$. Note that the asymptotic value of the force is attained as soon as the charge q is more than a few radii away from the sphere.

This example exhibits a general property that explains why an excess of charge on the surface does not immediately leave the surface because of mutual repulsion of the individual charges. As soon as an element of charge is removed from the surface, the image force tends to attract it back. If sufficient work is done, of course, charge can be removed from the surface to infinity. The work function of a metal is in large part just the work done against the attractive image force to remove an electron from the surface.

2.4 *Point Charge Near a Conducting Sphere at Fixed Potential*

Another problem that can be discussed easily is that of a point charge near a conducting sphere held at a fixed potential V. The potential is the same as for

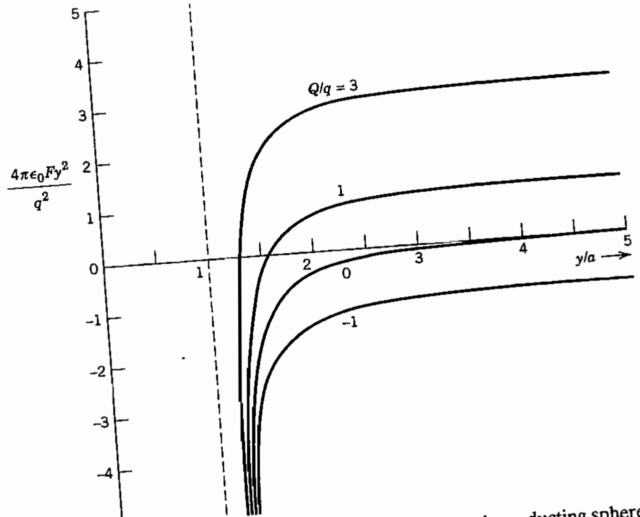

Figure 2.5 The force on a point charge q due to an insulated, conducting sphere of radius a carrying a total charge Q. Positive values mean a repulsion, negative an attraction. The asymptotic dependence of the force has been divided out. $4\pi\epsilon_0 Fy^2/q^2$ is plotted versus y/a for $Q/q = -1, 0, 1, 3$. Regardless of the value of Q, the force is always attractive at close distances because of the induced surface charge.

the charged sphere, except that the charge $(Q - q')$ at the center is replaced by a charge (Va). This can be seen from (2.8), since at $|\mathbf{x}| = a$ the first two terms cancel and the last term will be equal to V as required. Thus the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x} - \mathbf{y}|} - \frac{aq}{y\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|}\right] + \frac{Va}{|\mathbf{x}|} \tag{2.10}$$

The force on the charge q due to the sphere at fixed potential is

$$\mathbf{F} = \frac{q}{y^2}\left[Va - \frac{1}{4\pi\epsilon_0}\frac{qay^3}{(y^2 - a^2)^2}\right]\frac{\mathbf{y}}{y} \tag{2.11}$$

For corresponding values of $4\pi\epsilon_0 Va/q$ and Q/q this force is very similar to that of the charged sphere, shown in Fig. 2.5, although the approach to the asymptotic value (Vaq/y^2) is more gradual. For $Va \gg q$, the unstable equilibrium point has the equivalent location $y \simeq a(1 + \frac{1}{2}\sqrt{q/4\pi\epsilon_0 Va})$.

2.5 *Conducting Sphere in a Uniform Electric Field by Method of Images*

As a final example of the method of images we consider a conducting sphere of radius a in a uniform electric field E_0. A uniform field can be thought of as being

produced by appropriate positive and negative charges at infinity. For example, if there are two charges $\pm Q$, located at positions $z = \mp R$, as shown in Fig. 2.6a, then in a region near the origin whose dimensions are very small compared to R there is an approximately constant electric field $E_0 \simeq 2Q/4\pi\epsilon_0 R^2$ parallel to the z axis. In the limit as $R, Q \to \infty$, with Q/R^2 constant, this approximation becomes exact.

If now a conducting sphere of radius a is placed at the origin, the potential will be that due to the charges $\pm Q$ at $\mp R$ and their images $\mp Qa/R$ at $z = \mp a^2/R$:

$$\begin{aligned}\Phi = {} & \frac{Q/4\pi\epsilon_0}{(r^2 + R^2 + 2rR\cos\theta)^{1/2}} - \frac{Q/4\pi\epsilon_0}{(r^2 + R^2 - 2rR\cos\theta)^{1/2}} \\ & - \frac{aQ/4\pi\epsilon_0}{R\left(r^2 + \dfrac{a^4}{R^2} + \dfrac{2a^2r}{R}\cos\theta\right)^{1/2}} + \frac{aQ/4\pi\epsilon_0}{R\left(r^2 + \dfrac{a^4}{R^2} - \dfrac{2a^2r}{R}\cos\theta\right)^{1/2}}\end{aligned} \tag{2.12}$$

where Φ has been expressed in terms of the spherical coordinates of the observation point. In the first two terms R is much larger than r by assumption. Hence we can expand the radicals after factoring out R^2. Similarly, in the third and fourth terms, we can factor out r^2 and then expand. The result is:

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[-\frac{2Q}{R^2}r\cos\theta + \frac{2Q}{R^2}\frac{a^3}{r^2}\cos\theta\right] + \cdots \tag{2.13}$$

where the omitted terms vanish in the limit $R \to \infty$. In that limit $2Q/4\pi\epsilon_0 R^2$ becomes the applied uniform field, so that the potential is

$$\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{2.14}$$

The first term $(-E_0 z)$ is, of course, just the potential of a uniform field E_0 which could have been written down directly instead of the first two terms in (2.12).

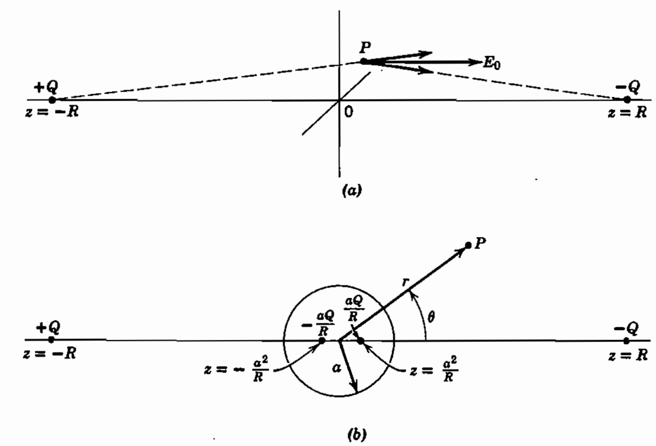

Figure 2.6 Conducting sphere in a uniform electric field by the method of images.

The second is the potential due to the induced surface-charge density or, equivalently, the image charges. Note that the image charges form a dipole of strength $D = Qa/R \times 2a^2/R = 4\pi\epsilon_0 E_0 a^3$. The induced surface-charge density is

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 3\epsilon_0 E_0 \cos\theta \tag{2.15}$$

We note that the surface integral of this charge density vanishes, so that there is no difference between a grounded and an insulated sphere.

2.6 *Green Function for the Sphere; General Solution for the Potential*

In preceding sections the problem of a conducting sphere in the presence of a point charge was discussed by the method of images. As mentioned in Section 1.10, the potential due to a unit source and its image (or images), chosen to satisfy homogeneous boundary conditions, is just the Green function (1.43 or 1.45) appropriate for Dirichlet or Neumann boundary conditions. In $G(\mathbf{x}, \mathbf{x}')$ the variable $\mathbf{x}'$ refers to the location P' of the unit source, while the variable $\mathbf{x}$ is the point P at which the potential is being evaluated. These coordinates and the sphere are shown in Fig. 2.7. For Dirichlet boundary conditions on the sphere of radius a the Green function defined via (1.39) for a unit source and its image is given by (2.1) with $q \rightarrow 4\pi\epsilon_0$ and relations (2.4). Transforming variables appropriately, we obtain the Green function:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x} - \dfrac{a^2}{x'^2}\mathbf{x}'\right|} \tag{2.16}$$

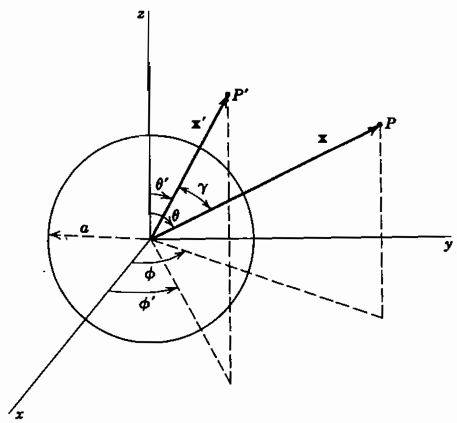

Figure 2.7

In terms of spherical coordinates this can be written:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\dfrac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{1/2}} \tag{2.17}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. The symmetry in the variables $\mathbf{x}$ and $\mathbf{x}'$ is obvious in the form (2.17), as is the condition that $G = 0$ if either $\mathbf{x}$ or $\mathbf{x}'$ is on the surface of the sphere.

For solution (1.44) of the Poisson equation we need not only G, but also $\partial G/\partial n'$. Remembering that $\mathbf{n}'$ is the unit normal outward from the volume of interest (i.e., inward along $\mathbf{x}'$ toward the origin), we have

$$\left.\frac{\partial G}{\partial n'}\right|_{x'=a} = -\frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax\cos\gamma)^{3/2}} \tag{2.18}$$

[Note that this is essentially the induced surface-charge density (2.5).] Hence the solution of the Laplace equation *outside* a sphere with the potential specified on its surface is, according to (1.44),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int \Phi(a, \theta', \phi') \frac{a(x^2 - a^2)}{(x^2 + a^2 - 2ax\cos\gamma)^{3/2}}\, d\Omega' \tag{2.19}$$

where $d\Omega'$ is the element of solid angle at the point (a, θ', ϕ') and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. For the *interior* problem, the normal derivative is radially outward, so that the sign of $\partial G/\partial n'$ is opposite to (2.18). This is equivalent to replacing the factor $(x^2 - a^2)$ by $(a^2 - x^2)$ in (2.19). For a problem with a charge distribution, we must add to (2.19) the appropriate integral in (1.44), with the Green function (2.17).

2.7 *Conducting Sphere with Hemispheres at Different Potentials*

As an example of the solution (2.19) for the potential outside a sphere with prescribed values of potential on its surface, we consider the conducting sphere of radius a made up of two hemispherical shells separated by a small insulating ring. The hemispheres are kept at different potentials. It will suffice to consider the potentials as $\pm V$, since arbitrary potentials can be handled by superposition of the solution for a sphere at fixed potential over its whole surface. The insulating ring lies in the $z = 0$ plane, as shown in Fig. 2.8, with the upper (lower) hemisphere at potential $+V$ $(-V)$.

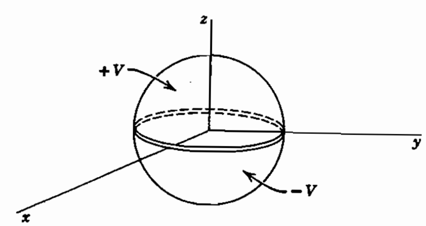

Figure 2.8